Fluid Dynamics by Chorlton

Unit-I

Kinematics of Fluids in motion

Real Fluids and Ideal fluids - velocity of a fluid at a point, stream lines, path lines, Steady and unsteady flows - Velocity Potential - The velocity vector - Local and particle rates of changes - Equation of continuity - worked examples - Acceleration of a fluid - condition at a rigid boundary.

Chap.: 2 , Sec.: 2.1 to 2.10

Fluid Mechanics
(the behaviour of the fluid (liquid or gases) at rest as well as in motion)

- Fluid Static (fluid at rest)
- Fluid Dynamics (fluid in motion, pressure forces are also consider)
- fluid Kinematics (fluid in motion, pressure forces are not considered)

Fluids
- liquids (incompressible), their volume do not change when the pressure changes
- gases (compressible),

hydrodynamics - moving incompressible fluids.

Ideal Fluid: A fluid which is incompressible and is having no viscosity, is known as ideal fluid.

Real Fluid: A fluid which is posses viscosity is known as real fluid.

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- Viscous fluid (It Satisfy a Newton Law of Viscosity)
- Inviscid fluid

Newton's Law of Viscosity

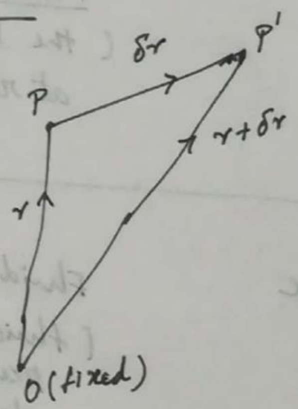
It is State Shear Stress (τ) on a fluid element layered. To the rate of Shear Stress. The constant of proportionality is called the coefficient of viscosity.

Mathematically

$$\tau \propto \frac{\partial u}{\partial y}$$

i.e. $$\tau = \mu \frac{\partial u}{\partial y}$$

Velocity of a Fluid at a Point



Let P be the position of a fluid particle at a time 't'

where $\overline{OP} = \bar{r}$

Let P' be the position of a fluid particle at a time $t+\Delta t$.

Then $\overline{OP'} = \bar{r} + \delta\bar{r}$

$$\therefore \overline{PP'} = \overline{OP'} - \overline{OP}$$
$$= \bar{r} + \delta\bar{r} - \bar{r}$$
$$= \delta\bar{r}$$

$\therefore$ The vector $\bar{q}$ of the fluid particle P is given by, $\bar{q} = \lim_{\delta t \to 0} \frac{\delta \bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$

$$= \frac{d}{dt}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\bar{q} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k} \longrightarrow (1)$$

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If (u, v, w) are the components of the velocity $(\bar{q})$ along the x, y, z direction respectively then

$$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k} \longrightarrow (2)$$

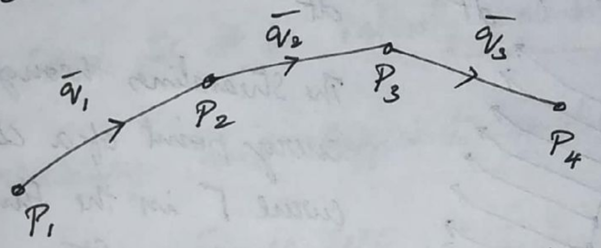
From (1) & (2), we obtained

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}$$

2.3 Stream line and path lines: Steady and unsteady flow

trajectories of material fluid elt.

We can draw a curve C in the fluid such that the direction of the tangent at a point $P(x, y, z)$ consider with the direction of $\bar{q}$ at P.



Then C is turned at stream lines, note that the stream lines are the solution of the differential equation.

where $\bar{q} = u\bar{i} + v\bar{j} + w\bar{k}$

i.e) $\bar{q}$ is parallel to $d\bar{r}$ and so $\bar{q} \times d\bar{r} = 0$

$$\therefore \quad \bar{q} \times d\bar{r} = 0$$

$$\because \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0 \Rightarrow \begin{array}{l} \bar{i}(v\,dz - w\,dy) - \bar{j}(u\,dz - w\,dz) \\ + \bar{k}(u\,dy - v\,dx) = 0 \end{array}$$

$$\therefore \; v\,dz - w\,dy = 0 \;, \; -u\,dz + w\,dz = 0 \;; \; u\,dy - v\,dx = 0$$

$$\frac{dz}{w} = \frac{dy}{v} \;; \quad \frac{dz}{w} = \frac{dx}{u} \;; \quad \frac{dy}{v} = \frac{dx}{u}$$

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(i.e) $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

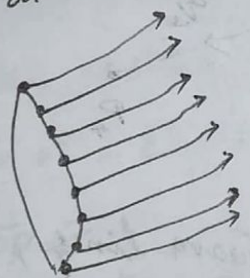
This is the differential equation of Stream line or line of flow.

Steady flows: when the motion is steady so that the pattern of flow not vary with time, the paths of the particles coincides with the Streamlines

unsteady flows: In unsteady motion, however, the flow pattern varies with time and the paths of the particles do not coincide with the Streamlines.

The pathlines are the Solutions of the differential equations

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$



The Streamlines through every point of a closed curve Γ in the fluid we obtain a Stream tube.

2.4 The velocity Potential & Velocity function

Suppose $\bar{q} = u\vec{i} + v\vec{j} + w\vec{k}$ is a velocity at any point $P(x,y,z)$ at time t.

Also Suppose that there exist a Scalar function $\phi(x,y,z,t)$. Uniform throughout the entire fluid flow Such that

$$-d\phi = u\,dx + v\,dy + w\,dz$$

$$-\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \frac{\partial\phi}{\partial t}dt\right) = u\,dx + v\,dy + w\,dz$$

Comparing, we have

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z}, \quad -\frac{\partial\phi}{\partial t} = 0 \longrightarrow (1)$$

hence, $\bar{q} = u\vec{i} + v\vec{j} + w\vec{k}$

$$= -\frac{\partial\phi}{\partial x}\vec{i} - \frac{\partial\phi}{\partial y}\vec{j} - \frac{\partial\phi}{\partial z}\vec{k}$$

$$= -\left[\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right]$$

In the equ. of $-\frac{\partial\phi}{\partial t} = 0$, declared that ϕ is a constant.

ie) ϕ is independent of t

Let $\phi = \phi(x, y, z, t)$

$\bar{q} = -\nabla\phi$ is the required equations, then

ϕ is said to ~~to~~ be velocity potential.

The necessary and sufficient condition for (1) to hold is curl $\bar{q}$ is zero. $[\nabla \times \bar{q} = 0] \longrightarrow (2)$

Irrotational (or) Potential Kind

The surface $\phi(x, y, z, t) = \text{constant}$ called equipotential. $\longrightarrow (3)$

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Equs. (1) & (2) show that at all points of the field of flow the equipotentials are cut orthogonally by the streamlines.

Example

Problems:

At the point in an incompressible fluid having Spherical polar coordinates (r, θ, ψ), the velocity components are $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, where M is a constant. Show that the velocity is the Potential Kind. Find the velocity potential and the equations of the Streamlines.

Solution:

The velocity of the potential is $\nabla \times \bar{q} = 0$

In Cartesian form of $\nabla \times \bar{q} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$

In polar form of $\nabla \times \bar{q} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \bar{a}_1 & h_2 \bar{a}_2 & h_3 \bar{a}_3 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}$

where $h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$

$\bar{a}_1 = \bar{r} \quad \bar{a}_2 = \bar{\theta} \quad \bar{a}_3 = \bar{\psi}$

$u = 2Mr^{-3}\cos\theta$

$v = Mr^{-3}\sin\theta$

$w = 0$

$$= \frac{1}{1 \cdot r \cdot r\sin\theta} \begin{vmatrix} r & r\theta & r\sin\theta\, \psi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ 2Mr^{-3}\cos\theta & Mr^{-3}\sin\theta \cdot r & 0 \end{vmatrix}$$

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$$= \frac{1}{r^2 \sin\theta}\left[r\left(0 - \frac{\partial}{\partial\psi}(Mr^{-3}\sin\theta)\right) - r\theta\left(0 - \frac{\partial}{\partial\psi}(2Mr^{-3}\cos\theta)\right) + r\sin\theta\,\psi\left(\frac{\partial}{\partial r}(Mr^{-2}\sin\theta) - \frac{\partial}{\partial\theta}(2Mr^{-3}\cos\theta)\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[0 - 0 + r\sin\theta\,\psi\,(-2Mr^{-3}\sin\theta + 2Mr^{-3}\sin\theta)\right]$$

$\nabla \times \bar{q} = 0$

$\therefore$ The flow is the Potential Kind

$\because$ the flow is Potential Kind, there exists a Velocity Potential ϕ Such that $\bar{q} = -\nabla\phi$

$$(2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0) = -\left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\right)$$

Then, $-\frac{\partial\phi}{\partial r} = 2Mr^{-3}\cos\theta$; $-\frac{\partial\phi}{\partial\theta}\cdot\frac{1}{r} = Mr^{-3}\sin\theta$;

$$-\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

$$\therefore \boxed{d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial\psi}d\psi}$$

$$= -(2Mr^{-3}\cos\theta)dr - (Mr^{-2}\sin\theta)d\theta + 0\,d\psi$$

Thus

$$\phi = \cancel{-}2M\cos\theta\left(\frac{r^{-2}}{\cancel{-2}}\right) + Mr^{-2}\cos\theta + 0$$

$$\therefore \boxed{\phi = Mr^{-2}\cos\theta + f(\theta)}$$

$$\phi = Mr^{-2}\cos\theta$$

The Stream lines are

$$h_1 \frac{dr}{v_1} = h_2 \frac{d\theta}{v_2} = h_3 \frac{d\psi}{v_3}$$

$$1. \quad \frac{dr}{2Mr^{-3}\cos\theta} = r.\frac{d\theta}{Mr^{-3}\sin\theta} = r\sin\theta . \frac{d\psi}{0}$$

$$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{d\theta}{Mr^{-4}\sin\theta} = \frac{r\sin\theta\, d\psi}{0}$$

3^{rd} ratio: $r\sin\theta\, d\psi = 0$ $\quad (\because dr = 0)$

$$d\psi = 0$$

Integrating, $\Rightarrow \boxed{\psi = constant}$

comparing the 1^{st} two ratio

$$\frac{dr}{2\cancel{M}\cancel{r^{-3}}\cos\theta} = \frac{r\,d\theta}{\cancel{M}\cancel{r^{-3}}\sin\theta}$$

$$\frac{dr}{r\cos\theta} = \frac{2\,d\theta}{\sin\theta}$$

$$\frac{dr}{r} = \frac{2\cos\theta\, d\theta}{\sin\theta}$$

Integrating, $\int \frac{dr}{r} = \int \frac{2\cos\theta}{\sin\theta}\, d\theta$

$$\log r = 2\log\sin\theta + \log c$$

$$= \log\sin^2\theta + \log c$$

$$\log r = \log(\sin^2\theta . c)$$

$$\boxed{r = c\sin^2\theta}$$

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2.5 The Vorticity vector (or) vorticity

We consider the flows for which $\text{Curl}\,\bar{q} = 0$

i.e. $\nabla \times \bar{q} = 0$

The vector $\xi = \nabla \times \bar{q}$ is called the vorticity vector. and its components are (ξ_1, ξ_2, ξ_3) given by

$$\xi_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \xi_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \xi_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The necessary and sufficient condition for Potential flow may be expressed by $\xi = 0$ $\quad (\because \nabla \times \bar{q} = 0)$

Vortex line:

A vortex lines is a curve drawn in the fluid such that the tangent to it at every point in the direction of the vorticity vector ξ.

In the cartesian components of ξ are $[\xi_1, \xi_2, \xi_3]$ the equation of the vortex lines are given by

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dz}{\xi_3}$$

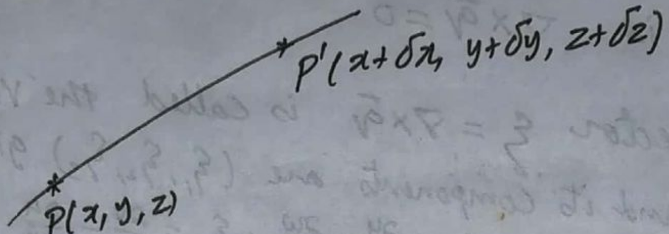
In general these do not coincide with the streamlines

Vortex Motion (or Rotational Motion): The fluid motion is said to be rotational if $\xi \neq 0 \Rightarrow \text{Curl}\,\bar{q} \neq 0$

Irrotational Motion: If $\xi = 0 \Rightarrow \text{Curl}\,\bar{q} = 0$, then the fluid motion is said to be irrotational (or) of potential kind and then $\bar{q} = -\nabla\phi$.

Vortex tube: It is the locus of vortex line drawn at each point of a closed curve i.e vortex tube is the surface formed by drawing vortex lines through each point of a closed curve in the fluid. A vortex tube with small cross-section is called a vortex filament.

2.6. Local and Particle Rates of Changes :



Suppose a particle of a fluid moves from $P(x,y,z)$ at time t to $P'(x+\delta x,\ y+\delta y,\ z+\delta z)$ at time $t+\delta t$

Let $f(x,y,z,t)$ be a scalar function associated with some property of the fluid (eg. density)

Then in the motion of the particle from P to P' the total change of f is given by,

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

Thus the total rate of change of 'f' at the point P at time 't' the motion of the particle is

$$\lim_{\delta t \to 0}\frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\lim_{\delta t\to 0}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\lim_{\delta t\to 0}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\lim_{\delta t\to 0}\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t}\lim_{\delta t\to 0}\frac{\delta t}{\delta t}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial z}w + \frac{\partial f}{\partial t}$$

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If $\bar{q} = [u, v, w]$ is the velocity of the fluid particle at P.

It can be written as

$$\frac{df}{dt} = \bar{q} \cdot \nabla f + \frac{\partial f}{\partial t} \longrightarrow (1)$$

|||ly, for a vector function $F(x, y, z, t)$ associated with some property of the fluid (Eg. velocity)

$$\frac{dF}{dt} = q \cdot \nabla F + \frac{\partial F}{\partial t} \longrightarrow (2)$$

For both, Scalar & vector functions we have established the operational equivalence.

$$\frac{d}{dt} \equiv q \cdot \nabla + \frac{\partial}{\partial t} \quad \text{(Substantial derivative)}$$

Applicable to both Scalar and Vector functions and it provided that these functions are associated with properties of the moving fluid.

Eqn. (1) & (2), we are considering the total changes in f or F when the fluid particles moves from $P(x, y, z)$ to $P'(x+\delta x, y+\delta y, z+\delta y)$ in time δt.

Thus $\frac{df}{dt}$ and $\frac{dF}{dt}$ are total time differentiations following the fluid particle and are called the Particle rates of change.

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On the otherhand the partial time derivative

$\frac{\partial f}{\partial t}$ and $\frac{\partial F}{\partial t}$ are only the time rates of change

at the point P. consider fixed in space,

they are the local rate of change.

Note:

$q.\nabla f$ (or) $q.\nabla F$ represents the rate of change to the motion of the particle along its path.

This point may also be the arc length of the path by s and PP' by δs.

Then if $\overline{PP'} \equiv \delta s\, \hat{s}$

$$\bar{q} = q\, \delta\hat{s}$$

where $q = |\bar{q}|$, and so

$$q.\nabla f = q\, \delta\hat{s}.\nabla f$$

$$= q \frac{\partial f}{\partial s}$$

111^{ly} for the vector function F. (using $\delta\hat{s}.\nabla \equiv \frac{\partial}{\partial s}$)

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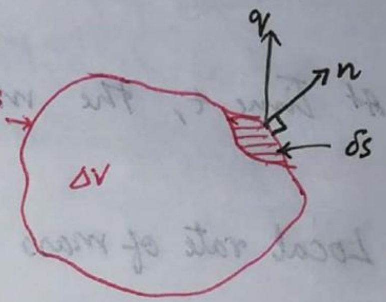
2.7. The Equation of Continuity by Euler's Method (Equation of Conservation of Mass)

When a region of a fluid contains neither sources nor sinks, that is to say when there are no inlets (source) or outlets (sinks) through which fluid can enter or leave the region, the amount of fluid within the region is conserved in accordance with the principle of conservation of matter. (equation of continuity)

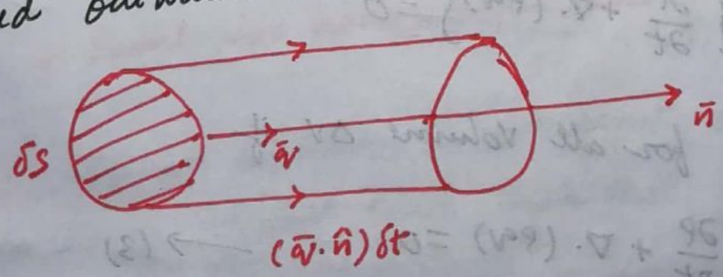


Let ΔS be a closed surface drawn in the fluid and fixed in space.

Suppose it contains a volume ΔV of the fluid.

Let $\rho = \rho(x, y, z, t)$ be the fluid density $\left(\rho = \frac{m}{V} = \frac{\text{mass}}{\text{volume}}\right)$ at any point (x, y, z) of the fluid in ΔV at any time 't'.

Suppose $\hat{n}$ is the unit outward-drawn normal at any surface element δS of ΔS, where $\delta S \ll \Delta S$. Then if $\bar{q}$ is the fluid velocity at the element δS, the normal component of $\bar{q}$ measured outwards from ΔV is $\hat{n}.\bar{q}$



Rate of efflux of fluid mass per unit time across $\delta S = \rho\, n.\bar{q}\,\delta S$.

Total rate of mass flow out of ΔV across $\Delta S = \int_{\Delta S} \rho\, n.q.\, ds$

Total rate of mass flow into $\Delta V = -\int_{\Delta S} n \cdot (\rho q)\, dS$

$$= -\int_{\Delta V} \nabla \cdot (\rho q)\, dv \longrightarrow (1)$$

$\left(\because \text{ by Gauss divergence thm} \right.$

$$\int_S \bar{F} \cdot \hat{n}\, dS = \int_V \nabla \cdot \bar{F}\, dv$$

At time t, the mass of fluid within the element ΔV $= \int_{\Delta V} \rho\, dv$.

$$\left.\begin{array}{l}\text{Local rate of mass increase within} \\ \Delta V\end{array}\right\} = \frac{\partial}{\partial t} \int_{\Delta V} \rho\, dv$$

$$= \int_{\Delta V} \frac{\partial \rho}{\partial t}\, dv \longrightarrow (2)$$

In the absence of sources and sinks within ΔV, matter (mass) is not created or destroyed in this region.

$\therefore$ Total rate of mass flow into ΔV $\overset{(1)}{=}$ Local rate of mass increase within ΔV (2)

$$-\int_{\Delta V} \nabla \cdot (\rho q)\, dV = \int_{\Delta V} \frac{\partial \rho}{\partial t}\, dv$$

$$\int_{\Delta V} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho q) \right\} = 0$$

It is true for all volume ΔV if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho q) = 0 \longrightarrow (3)$$

Equation (1) is the general equation of continuity

Cor 1.: $\nabla\cdot(\rho\bar{q}) = \rho\,\nabla\cdot\bar{q} + \nabla\rho\cdot q$

$$\therefore\ (3) \Rightarrow \quad \frac{\partial\rho}{\partial t} + \rho\,\nabla\cdot\bar{q} + \nabla\rho\cdot\bar{q} = 0 \longrightarrow (4)$$

Cor 2: W.K.T differential operator $\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{q}\cdot\nabla$

$$(4) \Rightarrow \quad \frac{d\rho}{dt} + \rho\,\nabla\cdot\bar{q} = 0 \longrightarrow (5)$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla\cdot\bar{q} = 0$$

$$\frac{d}{dt}(\log\rho) + \nabla\cdot\bar{q} = 0$$

Cor (3) When the motion of fluid is steady,

then $\frac{\partial\rho}{\partial t} = 0$ and thus

the equation of continuity (3) $\Rightarrow$ $\boxed{\nabla\cdot(\rho q) = 0}$

(here ρ is not a function of time, i.e $\rho = \rho(x, y, z)$)

Cor (4): When the fluid is incompressible, then ρ = constant

and thus

the equation of continuity (3) $\Rightarrow$ $\boxed{\nabla\cdot\bar{q} = 0}$

which is same for homogeneous and incompressible fluid.

Cor (5): If in addition to homogenity and incompressibility the flow is of potential kind such that $\boxed{\bar{q} = -\nabla\phi}$

then equ. (3) becomes

$$\operatorname{div}(-\nabla\phi) = 0$$

$$\Rightarrow \nabla\cdot(-\nabla\phi) = 0$$

$$\Rightarrow \boxed{\nabla^2\phi = 0}$$

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2.8 Worked Example

1) Test whether the motion specified by

$$\bar{q} = \frac{k^2(xj - yi)}{x^2+y^2} \quad (k = \text{constant})$$

is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

Sol.: Incompressible fluid:

i) First we have to prove $\nabla \cdot \bar{q} = 0$ (for an incompressible fluid)

$$\nabla \cdot \bar{q} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(-\frac{yK^2}{x^2+y^2}\vec{i} + \frac{K^2x}{x^2+y^2}\vec{j} + 0\vec{k}\right)$$

$$\left[\because \bar{q} = \frac{K^2(x\vec{j} - y\vec{i})}{x^2+y^2} = \frac{-yK^2\vec{i}}{x^2+y^2} + \frac{xK^2}{x^2+y^2}\vec{j} + 0\vec{k}\right.$$

$$= \frac{\partial}{\partial x}\left(-\frac{yK^2}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{K^2x}{x^2+y^2}\right) + \frac{\partial}{\partial z}(0)$$

$$= -yK^2\frac{\partial}{\partial x}((x^2+y^2)^{-1}) + K^2x\frac{\partial}{\partial y}((x^2+y^2)^{-1})$$

$$= -yK^2(-1(x^2+y^2)^{-2}(2x)) + K^2x(-1(x^2+y^2)^{-2}(2y))$$

$$= \frac{2K^2xy}{(x^2+y^2)^2} - \frac{2K^2xy}{(x^2+y^2)^2}$$

$$\boxed{\nabla \cdot \bar{q} = 0}$$

$\therefore$ The eqn. of continuity for an incompressible fluid is satisfied

(ii) Stream line:

The eqn. of stream line $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$$\frac{dx}{\frac{-K^2y}{x^2+y^2}} = \frac{dy}{\frac{K^2x}{x^2+y^2}} = \frac{dz}{0}$$

comparing the ratios (I) & (II), we obtain

$$\frac{dx}{\frac{-K^2y}{\cancel{x^2+y^2}}} = \frac{dy}{\frac{\cancel{K^2}x}{\cancel{x^2+y^2}}}$$

$$-x\,dx = y\,dy$$

$$x\,dx + y\,dy = 0$$

Integrating, $\int x\,dx + \int y\,dy = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c}{2}$$

$$\Rightarrow \boxed{x^2+y^2 = c_1}$$ streamline

III rd ratio, $\frac{dz}{0}$ $(\because dr = 0)$

$$dz = 0$$

Integrating $\int dz = 0$

$$\boxed{z = c_2}$$

(iii) Potential Kind To prove $\nabla \times \bar{q} = 0$

$$\nabla \times \bar{q} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-K^2y}{x^2+y^2} & \frac{K^2x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{K^2x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{K^2y}{x^2+y^2}\right)\right]$$

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$$= \bar{k}\left[\left(\frac{(x^2+y^2)k^2 - k^2x(2x)}{(x^2+y^2)^2}\right) + \left(\frac{(x^2+y^2)k^2 - k^2y(2y)}{(x^2+y^2)^2}\right)\right]$$

$$= \bar{k}\left[\frac{\cancel{x^2k^2} + \cancel{y^2k^2} - 2\cancel{k^2}x^2 + x^2\cancel{k^2} + y^2\cancel{k^2} - 2\cancel{k^2}y^2}{(x^2+y^2)^2}\right]$$

$$\boxed{\nabla \times \bar{q} = 0}$$

Thus the flow is of the potential kind and

(iv) we can find velocity potential $\phi(x,y,z)$

Such that $\bar{q} = -\nabla\phi$

$$u\bar{i} + v\bar{j} + w\bar{k} = -\left[\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right]$$

$$u = -\frac{\partial\phi}{\partial x} = -\frac{k^2y}{x^2+y^2} \; ; \; v = -\frac{\partial\phi}{\partial y} = \frac{k^2x}{x^2+y^2} \; ; \; w = -\frac{\partial\phi}{\partial z} = 0$$

$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \cancel{\frac{\partial\phi}{\partial z}}^{\,0}.dz$$

$$= \frac{k^2y}{x^2+y^2}dx - \frac{k^2x}{x^2+y^2}dy$$

$$= \frac{k^2(ydx - xdy)}{x^2+y^2}$$

$$d\phi = \frac{k^2 d(x/y)}{1+(\frac{x}{y})^2} = \frac{k^2dt}{1+t^2} \qquad \frac{x}{y} = t$$

Integrating, $\int d\phi = \int \frac{k^2dt}{1+t^2}$

$$\phi = k^2\tan^{-1}t + f(y)$$

$$\boxed{\phi(x,y) = k^2\tan^{-1}\left(\frac{x}{y}\right) + f(y)} \quad \text{—— (4)}$$

From this we find

$$\frac{\partial \phi}{\partial y} = f'(y) - \frac{K^2 x}{x^2+y^2}$$

$$\left[\frac{d}{dy}\left(\tan^{-1}(x/y)\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = \frac{-x\cancel{y^2}}{\cancel{y^2}(x^2+y^2)} = \frac{-x}{x^2+y^2} \right.$$

So comparing this with

$$\frac{\partial \phi}{\partial y} = \frac{-K^2 x}{x^2+y^2}, \text{ we get}$$

$$f'(y) - \frac{\cancel{K^2 x}}{\cancel{x^2+y^2}} = \frac{-\cancel{K^2 x}}{\cancel{x^2+y^2}}$$

$$f'(y) = 0$$

$$\boxed{f(y) = \text{constant}}$$

As the constant is immaterial, we take

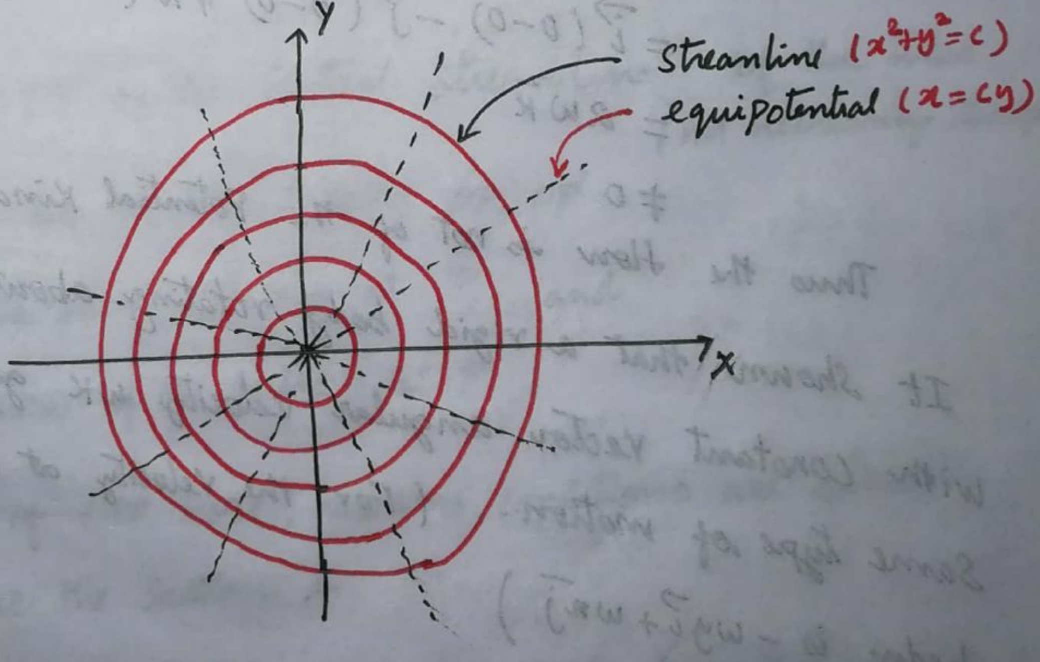
$$(*) \Rightarrow \quad \boxed{\phi(x,y) = K^2 \tan^{-1}(x/y)}$$

The equipotentials are thus given by the planes

$$\frac{x}{y} = c \quad \Rightarrow \quad \boxed{x = cy}$$

through the z-axis.

They are appropriately intersected orthogonally by the streamlines.



2) For an incompressible fluid, $\bar{q} = [-wy, wx, 0]$ (w = const.). Discuss the nature of the flow.

Sol.

i) Incompressible fluid — To prove $\nabla \cdot q = 0$

$$\nabla \cdot \bar{q} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)(-wy\bar{i} + wx\bar{j} + 0\bar{k})$$

$$= \frac{\partial}{\partial x}(-wy) + \frac{\partial}{\partial y}(wx) + \frac{\partial}{\partial z}(0)$$

$$\boxed{\nabla \cdot \bar{q} = 0}$$

$\therefore$ The eqn. of continuity for an incompressible fluid is satisfied

ii) Potential Kind — To prove $\nabla \times \bar{q} = 0$

$$\nabla \times \bar{q} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \bar{k}(w+w)$$

$$= 2w\bar{k}$$

$$\neq 0$$

Thus the flow is not of the Potential Kind.

It shown that a rigid body rotating about the z-axis with constant vector angular velocity $w\bar{k}$ gives the same type of motion. (For the velocity at (x,y,z) in the body is $-wy\vec{i} + wx\bar{j}$)

iii) Stream lines

The eqn. of the Streamlines are

$$\frac{dx}{-wy} = \frac{dy}{wx} = \frac{dz}{0}$$

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Compare First two ratio,

$$\frac{dx}{-\omega y} = \frac{dy}{\omega x}$$

$$-x\,dx = y\,dy$$

$$\boxed{x^2+y^2 = \text{constant}}$$

3rd ratio

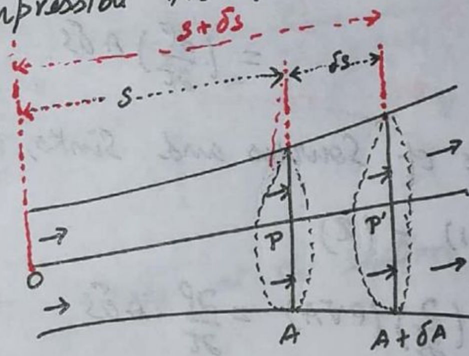
$$\frac{dz}{0} \qquad (\because dr=0)$$

$$dz = 0$$

$$\boxed{z = \text{constant}}$$

3) For a fluid motion in a fine tube of variable section A, prove from first principles that the equation of continuity is $A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}(A\rho v) = 0$ where v is the speed at a point P of the fluid and s the length of the tube upto P. What does this become for steady incompressible flow?

Sol.



Let OPP' be the central stream line of the tube (not necessarily straight)

Let s, $s+\delta s$ be the arc lengths OP, OP'

Let v be the fluid velocity at P and

A the area of the section at P

$\because$ tube is of fine bore, assume conditions are sensibly constant over the section A

so that the rate of mass flux over A in the sense of s increasing is $\rho v A$ per unit time.

At the neighbouring section A' through P', the mass flux per unit time in the direction of s increasing is

$$\therefore \rho V A + \delta s \frac{\partial}{\partial s}(\rho V A)$$

at the same instant of time t.

The net rate of flow of mass into the element between the section A, $A+\delta A$ (consider fixed in space) is $-\delta s \left(\frac{\partial}{\partial s}\right)(\rho V A) \longrightarrow (1)$

But at time t, the mass between the sections is $\rho A \delta s$.

The rate of increase is $\left(\frac{\partial}{\partial t}\right)(\rho A \delta s)$

$$= \left(\frac{\partial \rho}{\partial t}\right) A \delta s \longrightarrow (2)$$

In the absence of sources and sinks, then we get

$$(1) = (2)$$

$$-\delta s \left(\frac{\partial}{\partial s}\right)(\rho V A) = \frac{\partial \rho}{\partial t} . A \delta s$$

$$A \frac{\partial \rho}{\partial s} + \frac{\partial}{\partial s}(\rho V A) = 0 \longrightarrow (3)$$

For steady incompressible flow, ($\cdot \rho = \text{constant}$)

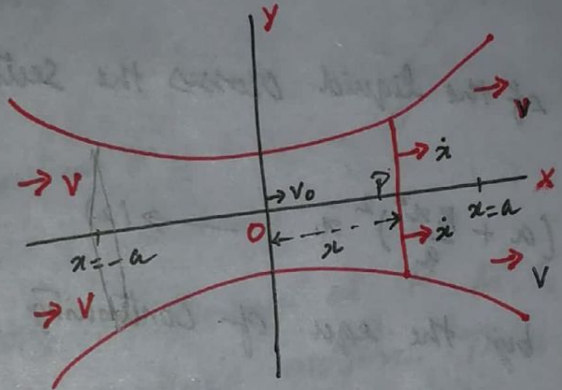
$$(3) \Rightarrow \frac{d}{ds}(VA) = 0$$

Integrating $VA = \text{constant}$ over every section.

$\therefore$ The volume of fluid crossing every section per unit time is constant.

4) Liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + \frac{Kx^2}{a}$ about the x-axis $(-a \le x \le a)$. If the liquid enters at the end $x = -a$ of the pipe with velocity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from $x = -a$ to $x = +a$ is $\frac{2a}{V(1+K)^2}\left(1 + \frac{2}{3}K + \frac{1}{5}K^2\right)$.

(Assume that k is so small that the flow remains appreciably 1-dimensional throughout)



Let V_0 be the velocity at the section $x = 0$ and

V be the velocity at $x = -a$

The surface of the revolution of the curve $y = a + \frac{Kx^2}{a}$ about the x-axis

Then the area of the section at $x = -a$ is

$$r = y = a + \frac{K(-a)^2}{a}$$

$$= a + Ka$$

$$= a(1+K)$$

$$\therefore \boxed{\pi r^2 = \pi a^2(1+K)^2}$$

Given $y = a + \frac{Kx^2}{a}$

$$\frac{(y-a)}{K} = x^2 \qquad \text{at } x = -a$$

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$$\frac{(y-a)\alpha}{\kappa} = a^{\alpha}$$

$$(y-a) = \kappa a$$

$$y = a(\kappa+1)$$

Consider the cross section at P of the pipe at a distance 'x' from 'O'.

This area of cross section $\boxed{\pi y^2 = \pi\left(a + \frac{\kappa x^2}{a}\right)^2}$

where $\dot{x}$ is the velocity of a fluid.

The volume of the liquid enters at $A = \pi a^2(1+\kappa)^2 V \rightarrow (1)$

The volume of the liquid crosses the section at P,

$$P = \pi\left(a + \frac{\kappa x^2}{a}\right)^2 \dot{x} \longrightarrow (2)$$

From (1) & (2) by the equ. of continuity (absence of Sources & Sink)

$$\pi a^2(1+\kappa)^2 V = \pi\left(a + \frac{\kappa x^2}{a}\right)^2 \dot{x}$$

$$\pi a^2(1+\kappa)^2 V = \pi\left(a + \frac{\kappa x^2}{a}\right)^2 \frac{dx}{dt}$$

$$\pi a^2(1+\kappa)^2 V \cdot dt = \pi\left(a + \frac{\kappa x^2}{a}\right)^2 dx$$

Integrating,

$$\int_0^T \pi a^2(1+\kappa)^2 V\, dt = \int_{-a}^{a} \pi\left(a + \frac{\kappa x^2}{a}\right)^2 dx$$

$$\cancel{\pi} a^2(1+\kappa)^2 V (t)_0^T = \int_{-a}^{a} \cancel{\pi}\left(a + \frac{\kappa x^2}{a}\right)^2 dx$$

$$= 2\int_0^a \cancel{\pi}\left(\frac{a^4 + \kappa^2 x^4 + 2a^2\kappa x^2}{a^2}\right) dx$$

$$a^4(1+\kappa)^2 VT = 2\left[a^4 x + \kappa^2\frac{x^5}{5} + 2a^2\kappa\frac{x^3}{3}\right]_0^a$$

$$= 2\left[a^4 \cdot a + \kappa^2\frac{a^5}{5} + 2a^2\kappa\frac{a^3}{3}\right]$$

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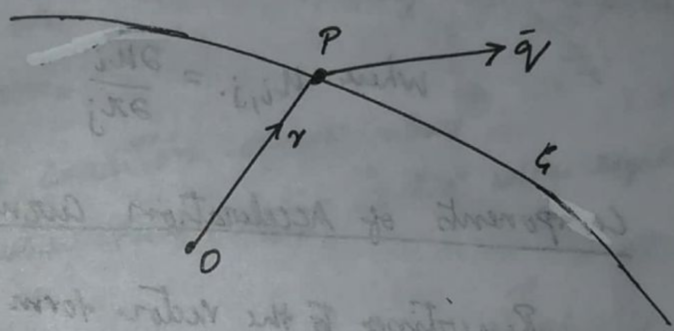
$$a^{4}(1+k)^2 V T = 2a^{5}\left(1+\frac{k^2}{5}+\frac{2\cdot k}{3}\right)$$

$$\boxed{T = \frac{2a\left(1+\frac{2k}{3}+\frac{k^2}{5}\right)}{V(1+k)^2}}$$

2.9 Acceleration of a Fluid



Let us consider Particle travelling along a curve C.

At time, its Position P is Specified by $\overline{OP} \equiv r$ and its velocity $\bar{q}$ along the tangent at P to C is in the direction of the particle's motion. Then the instantaneous acceleration $\bar{f}$ at P is

$$\bar{f} = \frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}. \longrightarrow (1)$$

Taking the Cartesian components of $\bar{q}$ as $[u, v, w]$, this shows that the components of acceleration are

$$\left.\begin{aligned} f_1 = \frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ f_2 = \frac{dv}{dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ f_3 = \frac{dw}{dt} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \longrightarrow (2)$$

which are the required Cartesian components of $\bar{f}$

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In tensor form, with coordinates x_i, and velocity components u_i $(i=1,2,3)$, the set of equ. (2) could be written as

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j\, u_{i,j}. \longrightarrow (3)$$

(* $i=1$, $j=1,2,3$; $\frac{du_1}{dt} = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3}$)

$u_1 = u$, $u_2 = v$, $u_3 = w$

$x_1 = x$, $x_2 = y$, $x_3 = z$

$|||^{ly}$ $i=2$, $i=3$)

where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$

Components of Acceleration curvilinear co-ordinates:

Reverting to the vector form (1), the term

$$(\bar{q}\cdot\nabla)\bar{q} = (\bar{q}\cdot\bar{i})\frac{\partial\bar{q}}{\partial x} + (\bar{q}\cdot\bar{j})\frac{\partial\bar{q}}{\partial y} + (\bar{q}\cdot\bar{k})\frac{\partial\bar{q}}{\partial z}$$

$$(\bar{q}\cdot\nabla)\bar{q} = \sum\left\{(\bar{q}\cdot\bar{i})\frac{\partial\bar{q}}{\partial x}\right\}$$

Since $\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$

i.e., $(\bar{A}\cdot\bar{B})\bar{C} = (\bar{A}\cdot\bar{C})\bar{B} - \bar{A}\times(\bar{B}\times\bar{C})$

taking $\bar{A} = \bar{q}$, $\bar{B} = \hat{i}$; $\bar{C} = \frac{\partial\bar{q}}{\partial x}$, we get

$$(\bar{q}\cdot\hat{i})\frac{\partial\bar{q}}{\partial x} = \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial x}\right)\hat{i} - \bar{q}\times\left(\bar{i}\times\frac{\partial\bar{q}}{\partial x}\right)$$

$$= \bar{i}\,\frac{\partial}{\partial x}\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\left(\bar{i}\times\frac{\partial\bar{q}}{\partial x}\right)$$

$$\therefore\ (\bar{q}\cdot\nabla)\bar{q} = \sum i\frac{\partial}{\partial x}\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\sum\left(i\times\frac{\partial\bar{q}}{\partial x}\right)$$

$$= \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q}) \longrightarrow (4)$$

Sub. (4) in (1) we get

$$f = \frac{d\bar{q}}{dt} = \frac{\partial\bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q}) \longrightarrow (5)$$

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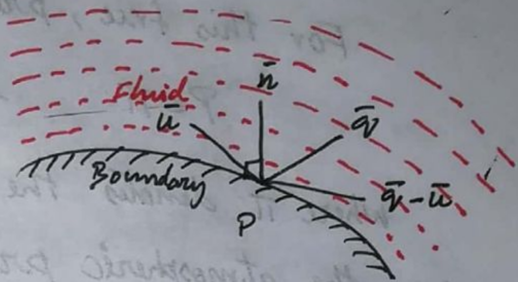
Equ. (5) useful for Potential flow for which $\nabla \times q = 0$

Equ. (5) may be useful that (1) for general orthogonal curvilinear coordinates.

2.10 Conditions at a Rigid Boundary



Physical conditions that should be satisfied on given boundaries of the fluid in motion, are called boundary conditions.

The simple boundary condition occurs where an ideal and incompressible fluid is in contact with rigid impermeable boundary.

Eg. wall of a container or the surface of a body which is moving through the fluid.

Let P be any point on the boundary surface where the velocity of fluid is $\bar{q}$ and velocity of the boundary surface is $\bar{u}$.

The velocity at the point of contact of the boundary surface and the liquid must be tangential to the surface otherwise the fluid will break it contact with the boundary surface. Thus, if $\hat{n}$ be the unit normal to the surface at the point of contact, then

$$(\bar{q}-\bar{u})\cdot\hat{n} = 0$$

$$\bar{q}\,\hat{n} = \bar{u}\,\hat{n} \longrightarrow (1)$$

In particular, if the boundary surface is at rest, the $\bar{u}=0$ and the condition becomes

$$\bar{q}\cdot\hat{n} = 0 \longrightarrow (2)$$

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Another type of boundary condition arrives at a free Surface where liquid borders a vacuum eg. the interface between liquid and air is usually regarded as free Surface.

For this free, pressure p Satisfies

$$P = \Pi \longrightarrow (3)$$

where Π denotes the pressure outside the fluid i.e. the atmospheric pressure.

Eqn. (3) is a dynamic boundary condition.

Third Type of boundary condition occurs at the boundary between two immiscible ideal fluids in which the velocities are $\bar{q}_1$ & $\bar{q}_2$ and pressure are p_1 & p_2 respectively.

To obtain the differential equation Satisfies ~~to be a~~ by boundary Surface of a fluid in motion

To find the condition that the Surface

$$F(\bar{r}, t) = F(x, y, z, t) = 0$$

may represent a boundary Surface.

If $\bar{q}$ be the velocity of fluid and u be the velocity of the boundary Surface at a point P of contact, then

$$(\bar{q} - \bar{u}) \cdot \hat{n} = 0$$

$$\Rightarrow \quad \bar{q} \cdot \hat{n} = \bar{u} \cdot \hat{n} \longrightarrow (1)$$

where $\bar{q} - \bar{u}$ is the relative velocity and $\hat{n}$ is a unit vector normal to the Surface at P.

where The equ. of the given Surface is

$$F(\bar{r},t) = F(x,y,z,t)=0 \longrightarrow (2)$$

W.K.T a unit vector normal to the Surface (2) is given by

$$\hat{n} = \frac{\nabla F}{|\nabla F|} \longrightarrow (3)$$

From(1), we get $\bar{q}\cdot\nabla F = \bar{u}\cdot\nabla F$

$\because$ the boundary Surface is itself in motion,

$\therefore$ at time $(t+\delta t)$, it's equ. is given by

$$F(\bar{r}+\delta\bar{r}, t+\delta t)=0 \longrightarrow (4)$$

From (2) & (4), we have

$$F(\bar{r}+\delta\bar{r},t+\delta t) - F(\bar{r},t)=0$$

(ie) $F(\bar{r}+\delta\bar{r}, t+\delta t) - F(\bar{r},t+\delta t) + F(\bar{r}, t+\delta t) - F(\bar{r},t)=0$

By Taylor's Series, we have

$$(\delta\bar{r}\cdot\nabla)F(\bar{r},t+\delta t) + \delta t\frac{\partial}{\partial t}F(\bar{r},t) = 0$$

$\left[\because F(x+\delta x, y+\delta y, z+\delta z) = F(x,y,z) + \delta x\frac{\partial F}{\partial x} + \delta y\frac{\partial F}{\partial y} + \delta z\frac{\partial F}{\partial z} + \ldots\right.$

$\left. = F(x,y,z) + \delta\bar{r}\cdot\nabla F\right]$

$$\Rightarrow \left(\frac{\delta\bar{r}}{\delta t}\cdot\nabla\right)F(\bar{r},t+\delta t) + \frac{\partial F}{\partial t} = 0$$

Taking limit as $\delta t\to 0$, we get

$$\lim_{\delta t\to 0}\left(\frac{\delta\bar{r}}{\delta t}\cdot\nabla\right)F + \frac{\partial F}{\partial t} = 0$$

$$\Rightarrow \frac{\partial F}{\partial t} + (\bar{q}\cdot\nabla)F = 0$$

i.e) $\boxed{\dfrac{dF}{dt}=0} \longrightarrow (5)$ $\quad \left(\because \dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \bar{q}\cdot\nabla\right)$

which is the required condition for any Surface F to be a boundary Surface.

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Cor. 1: If $\bar{q} = (u, v, w)$, then the condition

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

Suppose, the surface is rigid and does not move with time, then $\frac{\partial F}{\partial t} = 0$ (steady state) and the boundary condition is $u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$

Cor (2): The boundary condition

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

is the linear equation and its solution gives

$$\frac{dt}{1} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$\frac{D}{Dt} \equiv \frac{d}{dt}$ in Lagrangian view

$$\Rightarrow \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

which are the equations of path lines

Hence once a particle is in contact with the surface, it never leaves the surface.

Cor 3: From eqn. (5), we have

$$\bar{q} \cdot \nabla F = -\frac{\partial F}{\partial t}$$

$$\bar{q} \cdot \frac{\nabla F}{|\nabla F|} = \frac{-\partial F/\partial t}{|\nabla F|}$$

$$\bar{q} \cdot \hat{n} = \frac{-\partial F/\partial t}{|\nabla F|}$$

which gives the normal velocity.

Also from (1), we get

$$\bar{u} \cdot \hat{n} = \frac{-\partial F/\partial t}{|\nabla F|} \qquad [\because \bar{q}\,\hat{n} = \bar{u} \circ \hat{n}]$$

which gives the normal velocity of the boundary surface